The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy

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Abstract

The sixth Painlevé equation arises from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

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Introduction

The Drinfeld-Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy [DS]. It is known that their similarity reductions imply several Painlevé equations [AS, KK1, NY1]. For the sixth Painlevé equation (P_{VI}), the relation with the $A_2^{(1)}$-type hierarchy is investigated [KK2]. On the other hand, P_{VI} admits a group of symmetries which is isomorphic to the affine Weyl group of type $D_4^{(1)}$ [O]. Also it is known that P_{VI} is derived from the Lax pair associated with the algebra $\widehat{\mathfrak{so}}(8)$ [NY3]. However, the relation between $D_4^{(1)}$-type hierarchies and P_{VI} has not been clarified. In this paper, we show that the sixth Painlevé equation is derived from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

Consider a Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \tag{0.1}$$

with the Riemann scheme

$$\left\{ \begin{array}{cccccc} x = t_0 & x = t_1 & x = t_3 & x = t_4 & x = \lambda & x = \infty \\ 0 & 0 & 0 & 0 & 0 & \rho \\ \theta_0 & \theta_1 & \theta_3 & \theta_4 & 2 & \rho + 1 \end{array} \right\},$$

satisfying the relation

$$\theta_0 + \theta_1 + \theta_3 + \theta_4 + 2\rho = 1.$$

We also let $\mu = \mathrm{Res}_{x=\lambda}\, p_2(x)dx$. Then the monodromy preserving deformation of the equation (0.1) is described as a system of partial differential equations for λ and μ. This system can be regarded as the symmetric representation of P_{VI} [Kaw]. We discuss a derivation of the symmetric representation in the case

$$\begin{aligned} &t_0 = -t \quad t_1 = -\frac{t+1}{t-1} \quad t_3 = \frac{t-1}{t+1} \quad t_4 = \frac{1}{t} \\ &\theta_0 = \alpha_0 \quad \theta_1 = \alpha_1 - 1 \quad \theta_3 = \alpha_3 - 1 \quad \theta_4 = \alpha_4 - 1 \quad \rho = \alpha_2. \end{aligned}$$

Note that

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 4.$$

With the notation

$$F_0 = \lambda + t, \quad F_1 = \lambda + \frac{t+1}{t-1}, \quad F_2 = \mu, \quad F_3 = \lambda - \frac{t-1}{t+1}, \quad F_4 = \lambda - \frac{1}{t},$$

the dependence of λ and μ on t is given by

$$\begin{aligned} \vartheta(F_j) = {}& 2F_0F_1F_2F_3F_4 - (\alpha_0 - 1)F_1F_3F_4 \\ & - (\alpha_1 - 1)F_0F_3F_4 - (\alpha_3 - 1)F_0F_1F_4 - (\alpha_4 - 1)F_0F_1F_3 + \Theta_j, \end{aligned} \tag{0.2}$$

for $j = 0, 1, 3, 4$ and

$$\begin{aligned} \vartheta(F_2) = {}& -F_2^2(F_0F_1F_3 + F_0F_1F_4 + F_0F_3F_4 + F_1F_3F_4) \\ & + F_2\{(\alpha_3 + \alpha_4 - 2)F_0F_1 + (\alpha_1 + \alpha_4 - 2)F_0F_3 + (\alpha_1 + \alpha_3 - 2)F_0F_4 \\ & + (\alpha_0 + \alpha_4 - 2)F_1F_3 + (\alpha_0 + \alpha_3 - 2)F_1F_4 + (\alpha_0 + \alpha_1 - 2)F_3F_4\} \\ & - \alpha_2\{(\alpha_0 + \alpha_2 - 1)F_0 + (\alpha_1 + \alpha_2 - 1)F_1 + (\alpha_3 + \alpha_2 - 1)F_3 \\ & + (\alpha_4 + \alpha_2 - 1)F_4\}, \end{aligned} \tag{0.3}$$

where

$$\vartheta = \Theta_0 \frac{d}{dt}, \quad \Theta_i = \prod_{j=0,1,3,4;j\neq i} (F_i - F_j).$$

Note that the system (0.2), (0.3) is equivalent to the Hamiltonian system:

$$\frac{d\lambda}{dt} = \frac{\partial H'}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H'}{\partial \lambda}, \tag{0.4}$$

where the Hamiltonian $H' = H'(\lambda, \mu, t)$ is given by

$$\begin{aligned}\Theta_0 H' = & F_0F_1F_2^2F_3F_4 - (\alpha_0 - 1)F_1F_2F_3F_4 - (\alpha_1 - 1)F_0F_2F_3F_4 \\ & - (\alpha_3 - 1)F_0F_1F_2F_4 - (\alpha_4 - 1)F_0F_1F_2F_3 + \alpha_2F_0\{(\alpha_0 - 1)F_0 \\ & + (\alpha_1 + \alpha_2 - 1)F_1 + (\alpha_3 + \alpha_2 - 1)F_3 + (\alpha_4 + \alpha_2 - 1)F_4\}.\end{aligned}$$

We also remark that the system (0.4) is transformed into the Hamiltonian system for P_{VI} as in [IKSY]

$$\frac{dq}{ds} = \frac{\partial H}{\partial p}, \quad \frac{dp}{ds} = -\frac{\partial H}{\partial q},$$

with the Hamiltonian

$$\begin{aligned}s(s-1)H = & q(q-1)(q-s)p^2 - \frac{1}{4}\{(\alpha_1 - 4)q(q-1) \\ & + \alpha_3 q(q-s) + \alpha_4(q-1)(q-s)\}p + \frac{1}{16}\alpha_2(\alpha_0 + \alpha_2)q,\end{aligned}$$

by the canonical transformation $(\lambda, \mu, t, H') \to (q, p, s, H)$ defined as

$$q = \frac{(t + \frac{t-1}{t+1})F_4}{(\frac{t-1}{t+1} - \frac{1}{t})F_0}, \quad p = \frac{(\frac{t-1}{t+1} - \frac{1}{t})F_0(F_0F_2 + \alpha_2)}{4(t + \frac{t-1}{t+1})(t + \frac{1}{t})},$$

and

$$s = -\frac{(t + \frac{t-1}{t+1})(\frac{t+1}{t-1} + \frac{1}{t})}{(t - \frac{t+1}{t-1})(\frac{t-1}{t+1} - \frac{1}{t})}.$$

This paper is organized as follows. In Section 1, we recall the definition of the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(D_4^{(1)})$. In Section 2, a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ is formulated. In Sections 3 and 4, we show that its similarity reduction implies the symmetric representation of P_{VI}.

1 Affine Lie algebra

In the notation of [Kac], the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(D_4^{(1)})$ is the Lie algebra generated by the Chevalley generators e_i, f_i, $\alpha_i^\vee$ $(i = 0, \dots, 4)$ and the scaling element d with the fundamental relations

$$\begin{aligned}& (\mathrm{ad}e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad}f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j), \\ & [\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_j] = a_{ij}e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \\ & [d, \alpha_i^\vee] = 0, \quad [d, e_i] = \delta_{i,0}e_0, \quad [d, f_i] = -\delta_{i,0}f_0,\end{aligned}$$

for $i,j = 0,\dots,4$, where $A = (a_{ij})_{i,j=0}^4$ is the generalized Cartan matrix of type $D_4^{(1)}$ defined by

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$\mathfrak{h} = \bigoplus_{j=0}^{4} \mathbb{C}\alpha_j^\vee \oplus \mathbb{C}d.$$

The canonical central element of $\mathfrak{g}$ is given by

$$K = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee.$$

The normalized invariant form $(\,|\,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$\begin{aligned} &(\alpha_i^\vee|\alpha_j^\vee) = a_{ij}, \quad &&(e_i|f_j) = \delta_{i,j}, \quad &&(\alpha_i^\vee|e_j) = (\alpha_i^\vee|f_j) = 0, \\ &(d|d) = 0, &&(d|\alpha_j^\vee) = \delta_{0,j}, &&(d|e_j) = (d|f_j) = 0, \end{aligned}$$

for $i,j = 0,\dots,4$.

We consider the $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(s)$ of type $s = (1,1,0,1,1)$ by setting

$$\begin{aligned} &\deg \mathfrak{h} = \deg e_2 = \deg f_2 = 0, \\ &\deg e_i = 1, \quad \deg f_i = -1 \quad (i = 0,1,3,4). \end{aligned}$$

If we take an element $d_s \in \mathfrak{h}$ such that

$$(d_s|\alpha_2^\vee) = 0, \quad (d_s|\alpha_j^\vee) = 1 \quad (j = 0,1,3,4),$$

this gradation is defined by

$$\mathfrak{g}_k(s) = \{x \in \mathfrak{g} \mid [d_s, x] = kx\} \quad (k \in \mathbb{Z}).$$

In the following, we choose

$$d_s = 4d + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee.$$

We set

$$\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k(s), \quad \mathfrak{g}_{\geq 0} = \bigoplus_{k\geq 0} \mathfrak{g}_k(s).$$

We choose the graded Heisenberg subalgebra $\mathfrak{s} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{s}_k(s)$ of $\mathfrak{g}$ of type $s = (1,1,0,1,1)$ with

$$\mathfrak{s}_1(s) = \mathbb{C}\Lambda_{1,1} \oplus \mathbb{C}\Lambda_{1,2},$$

where

$$\begin{aligned}\Lambda_{1,1} &= -e_0 + e_1 + e_3 - e_{21} + e_{23} + e_{24},\\ \Lambda_{1,2} &= e_1 - e_3 + e_4 + e_{20} + e_{21} + e_{23}.\end{aligned}$$

Here we denote

$$e_{2j} = [e_2, e_j], \quad f_{2j} = [f_2, f_j] \quad (j = 0,1,3,4).$$

We remark that

$$\mathfrak{s} = \left\{ x \in \mathfrak{g} \;\middle|\; [\Lambda_{1,1}, x] \in \mathbb{C}K \right\}.$$

and

$$\mathfrak{s}_0(s) = \mathbb{C}K, \quad \mathfrak{s}_{2k}(s) = 0 \quad (k \neq 0).$$

Each $\mathfrak{s}_{2k-1}(s)$ is expressed in the form

$$\mathfrak{s}_{2k-1}(s) = \mathbb{C}\Lambda_{2k-1,1} \oplus \mathbb{C}\Lambda_{2k-1,2},$$

with certain elements $\Lambda_{2k-1,i}$ $(i = 1,2)$ satisfying

$$[\Lambda_{2k-1,i}, \Lambda_{2l-1,j}] = (2k-1)\delta_{i,j}\delta_{k+l,1}K \quad (i,j = 1,2; k,l \in \mathbb{Z}).$$

For $k = 0$, we have

$$\begin{aligned}\Lambda_{-1,1} &= \frac{1}{2}(-2f_0 + f_1 + f_3 + f_{21} - f_{23} - 2f_{24}),\\ \Lambda_{-1,2} &= \frac{1}{2}(f_1 - f_3 + 2f_4 - 2f_{20} - f_{21} - f_{23}).\end{aligned}$$

Remark 1.1. *In the nortation of* [C]*, the Heisenberg subalgebra* $\mathfrak{s}$ *corresponds to the conjugacy class* $D_4(a_1)$ *of the Weyl group* $W(D_4)$*; see* [DF]*.*

2 Drinfeld-Sokolov hierarchy

In the following, we use the notation of infinite dimensional groups

$$G_{<0} = \exp(\widehat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\widehat{\mathfrak{g}}_{\geq 0}),$$

where $\widehat{\mathfrak{g}}_{<0}$ and $\widehat{\mathfrak{g}}_{\geq 0}$ are completions of $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geq 0}$ respectively.

Introducing the time variables $t_{k,i}$ $(i=1,2; k=1,3,5,\ldots)$, we consider the *Sato equation* for a $G_{<0}$-valued function $W = W(t_{1,1}, t_{1,2}, \ldots)$

$$\partial_{k,i}(W) = B_{k,i}W - W\Lambda_{k,j} \quad (i=1,2; k=1,3,5,\ldots), \tag{2.1}$$

where $\partial_{k,i} = \partial/\partial t_{k,i}$ and $B_{k,i}$ stands for the $\mathfrak{g}_{\geq 0}$-component of $W\Lambda_{k,i}W^{-1} \in \widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. We understand the Sato equation (2.1) as a conventional form of the differential equation

$$\partial_{k,i} - B_{k,i} = W(\partial_{k,i} - \Lambda_{k,i})W^{-1} \quad (i=1,2; k=1,3,5,\ldots), \tag{2.2}$$

defined through the adjoint action of $G_{<0}$ on $\widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. The Zakharov-Shabat equation

$$[\partial_{k,i} - B_{k,i}, \partial_{l,j} - B_{l,j}] = 0 \quad (i,j=1,2; k,l=1,3,5,\ldots), \tag{2.3}$$

follows from the Sato equation (2.2).

The $\mathfrak{g}_{\geq 0}$-valued functions $B_{1,i}$ $(i=1,2)$ are expressed in the form

$$B_{1,i} = \Lambda_{1,i} + U_i, \quad U_i = \sum_{j=0}^{4} u_{j,i}\alpha_j^{\vee} + x_i e_2 + y_i f_2. \tag{2.4}$$

The Zakharov-Shabat equation (2.3) for $k=1$ is equivalent to

$$\partial_{1,i}(U_j) - \partial_{1,j}(U_i) + [U_j, U_i] = 0, \quad [\Lambda_{1,i}, U_j] - [\Lambda_{1,j}, U_i] = 0, \tag{2.5}$$

for $i,j=1,2$. Then we have

Lemma 2.1. *Under the Sato equation* (2.2), *the following equations are satisfied*:

$$(d_s|\partial_{1,i}(U_j)) + \frac{1}{2}(U_i|U_j) = 0 \quad (i,j=1,2). \tag{2.6}$$

Proof. The system (2.2) for $k=1$ is equivalent to

$$\partial_{1,i} - \Lambda_{1,i} - U_i = W(\partial_{1,i} - \Lambda_{1,i})W^{-1} \quad (i=1,2). \tag{2.7}$$

Set

$$W = \exp(w), \quad w = \sum_{k=1}^{\infty} w_{-k}, \quad w_{-k} \in \mathfrak{g}_{-k}(s).$$

Then the system (2.7) implies

$$U_i = \sum_{k=1}^{\infty} \frac{1}{k!}\mathrm{ad}(w)^{k-1}\partial_{1,i}(w) + \sum_{k=1}^{\infty} \frac{1}{k!}\mathrm{ad}(w)^{k}(\Lambda_{1,i}) \quad (i=1,2). \tag{2.8}$$

Comparing the component of degree $-k$ in (2.8), we obtain

$$U_i = \mathrm{ad}(w_{-1})(\Lambda_{1,i}) \quad (i = 1, 2),$$

for $k = 0$;

$$\mathrm{ad}(w_{-2})(\Lambda_{1,i}) + \frac{1}{2}\mathrm{ad}(w_{-1})^2(\Lambda_{1,i}) + \partial_{1,i}(w_{-1}) = 0 \quad (i = 1, 2), \tag{2.9}$$

for $k = 1$;

$$\sum_{i_1+\ldots+i_l=k+1} \frac{1}{l!}\mathrm{ad}(w_{-i_1})\ldots\mathrm{ad}(w_{-i_l})(\Lambda_{1,i}) \\ + \sum_{i_1+\ldots+i_l=k} \frac{1}{l!}\mathrm{ad}(w_{-i_1})\ldots\mathrm{ad}(w_{-i_{l-1}})\partial_{1,i}(w_{-i_l}) = 0 \quad (i = 1, 2),$$

for $k \geq 2$. On the other hand, we have

$$(\Lambda_{1,i}|\mathrm{ad}(\Lambda_{1,j})(x)) = 0 \quad (i, j = 1, 2; x \in \mathfrak{g}_{-2}(s)),$$

and

$$(\Lambda_{1,i}|x) = (d_s|\mathrm{ad}(\Lambda_{1,i})(x)) \quad (i = 1, 2; x \in \mathfrak{g}_{-1}(s)).$$

Hence it follows that

$$\begin{aligned}(\Lambda_{1,j}|\text{LHS of (2.9)}) &= \frac{1}{2}(\Lambda_{1,j}|\mathrm{ad}(w_{-1})^2(\Lambda_{1,i})) + (\Lambda_{1,j}|\partial_{1,i}(w_{-1})) \\ &= -\frac{1}{2}(U_i|U_j) - (d_s|\partial_{1,i}(U_j)).\end{aligned}$$

□

Remark 2.2. *Let $X(0) \in G_{<0}G_{\geq 0}$ and define*

$$X = X(t_{1,1}, t_{1,2}, \ldots) = \exp(\xi)X(0), \quad \xi = \sum_{i=1,2}\sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$

Then a solution $W \in G_{<0}$ of the system (2.1) is given formally via the decomposition

$$X = W^{-1}Z, \quad Z \in G_{\geq 0}.$$

3 Similarity reduction

Under the Sato equation (2.2), we consider the operator

$$\mathcal{M} = W \exp(\xi) d_s \exp(-\xi) W^{-1}, \quad \xi = \sum_{i=1,2} \sum_{k=1,3,\ldots} t_{k,i} \Lambda_{k,i}.$$

Then the operator $\mathcal{M}$ satisfies

$$\partial_{k,i}(\mathcal{M}) = [B_{k,i}, \mathcal{M}] \quad (i = 1, 2; k = 1, 3, 5, \ldots).$$

Note that

$$\mathcal{M} = d_s - \sum_{i=1,2} \sum_{k=1,3,\ldots} k t_{k,i} W \Lambda_{k,i} W^{-1} - d_s(W) W^{-1}.$$

Assuming that $t_{k,1} = t_{k,2} = 0$ for $k \geq 3$, we require that the similarity condition $\mathcal{M} \in \mathfrak{g}_{\geq 0}$ is satisfied. Then we have

$$\partial_{1,i}(\mathcal{M}) = [B_{1,i}, \mathcal{M}] \quad (i = 1, 2).$$

where $\mathcal{M} = d_s - t_{1,1} B_{1,1} - t_{1,2} B_{1,2}$, or equivalently

$$[d_s - M, \partial_{1,i} - B_{1,i}] = 0 \quad (i = 1, 2), \tag{3.1}$$

where $M = t_{1,1} B_{1,1} + t_{1,2} B_{1,2}$. Under the Zakharov-Shabat equation

$$[\partial_{1,1} - B_{1,1}, \partial_{1,2} - B_{1,2}] = 0,$$

the system (3.1) is equivalent to

$$\sum_{j=1,2} t_{1,j} \partial_{1,j}(B_{1,i}) = [d_s, B_{1,i}] - B_{1,i} \quad (i = 1, 2).$$

In terms of the operators U_i, this similarity condition can be expressed as

$$\sum_{j=1,2} t_{1,j} \partial_{1,j}(U_i) + U_i = 0 \quad (i = 1, 2). \tag{3.2}$$

We regard the systems (2.5), (2.6) and (3.2) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$.

In the notation (2.4), these systems are expressed in terms of the variables $u_{j,i}$, x_i, y_i as follows:

$$\begin{aligned}
&\partial_{1,1}(x_2) - \partial_{1,2}(x_1) \\
&\quad - (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2}) x_1 + (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2}) x_2 = 0, \\
&\partial_{1,1}(y_2) - \partial_{1,2}(y_1) \\
&\quad + (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2}) y_1 - (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2}) y_2 = 0, \\
&\partial_{1,1}(u_{2,2}) - \partial_{1,2}(u_{2,1}) - x_1 y_2 + x_2 y_1 = 0, \\
&\partial_{1,1}(u_{j,2}) - \partial_{1,2}(u_{j,1}) = 0 \quad (j = 0, 1, 3, 4),
\end{aligned}$$

and

$$\begin{aligned}
&u_{1,1} - 2u_{2,1} + u_{3,1} + 2u_{4,1} - u_{1,2} + u_{3,2} = 0,\\
&u_{1,1} - u_{3,1} - 2u_{0,2} - u_{1,2} + 2u_{2,2} - u_{3,2} = 0,\\
&u_{1,1} - u_{3,1} + u_{1,2} + u_{3,2} - 2u_{4,2} + 2x_1 = 0,\\
&2u_{0,1} - u_{1,1} - u_{3,1} - u_{1,2} + u_{3,2} + 2x_2 = 0,\\
&u_{1,1} - u_{3,1} + 2u_{0,2} - u_{1,2} - u_{3,2} + 2y_1 = 0,\\
&u_{1,1} + u_{3,1} - 2u_{4,1} - u_{1,2} + u_{3,2} + 2y_2 = 0,
\end{aligned} \tag{3.3}$$

for the system (2.5);

$$\begin{aligned}
&\sum_{l=0,1,3,4} 4\partial_{1,i}(u_{l,j})\\
&\quad + \sum_{l=0,1,3,4} (2u_{l,i} - u_{2,i})(2u_{l,j} - u_{2,j}) + 2(x_i y_j + y_i x_j) = 0 \quad (i, j = 1, 2),
\end{aligned}$$

for the system (2.6);

$$\begin{aligned}
&t_{1,1}\partial_{1,1}(x_i) + t_{1,2}\partial_{1,2}(x_i) + x_i = 0, \quad t_{1,1}\partial_{1,1}(y_i) + t_{1,2}\partial_{1,2}(y_i) + y_i = 0,\\
&t_{1,1}\partial_{1,1}(u_{j,i}) + t_{1,2}\partial_{1,2}(u_{j,i}) + u_{j,i} = 0, \quad (i = 1, 2; j = 0, \ldots, 4),
\end{aligned}$$

for the system (3.2). In the next section, we show that they imply the sixth Painlevé equation.

Under the similarity condition (3.2), the system (2.6) implies

$$2(d_s|U_i) - t_{1,1}(U_i|U_1) - t_{1,2}(U_i|U_2) = 0 \quad (i = 1, 2).$$

It is expressed in terms of the variables $u_{j,i}$, x_i, y_i as follows:

$$\begin{aligned}
&\sum_{l=0,1,3,4} 4u_{l,i} - \sum_{l=0,1,3,4} t_{1,1}(2u_{l,i} - u_{2,i})(2u_{l,1} - u_{2,1}) - 2t_{1,1}(x_i y_1 + y_i x_1)\\
&\quad - \sum_{l=0,1,3,4} t_{1,2}(2u_{l,i} - u_{2,i})(2u_{l,2} - u_{2,2}) - 2t_{1,2}(x_i y_2 + y_i x_2) = 0 \quad (i = 1, 2).
\end{aligned} \tag{3.4}$$

Remark 3.1. *The systems* (2.5) *and* (3.2) *can be regarded as the compatibility condition of the Lax form*

$$d_s(\Psi) = M\Psi, \quad \partial_{1,i}(\Psi) = B_{1,i}\Psi \quad (i = 1, 2), \tag{3.5}$$

where $\Psi = W\exp(\xi)$*.*

4 The sixth Painlevé equation

In the previous section, we have derived the system of the equations

$$\begin{aligned}&\partial_{1,i}(U_j)-\partial_{1,j}(U_i)+[U_j,U_i]=0,\quad [\Lambda_{1,i},U_j]-[\Lambda_{1,j},U_i]=0,\\&(d_s|\partial_{1,i}(U_j))-\frac{1}{2}(U_i|U_j)=0,\quad \sum_{l=1,2}t_{1,l}\partial_{1,l}(U_i)+U_i=0\quad (i,j=1,2),\end{aligned}\tag{4.1}$$

for the $\mathfrak{g}_0$-valued functions $U_i=U_i(t_{1,1},t_{1,2})$ $(i=1,2)$, as a similarity reduction of the $D_4^{(1)}$ hierarchy of type $s=(1,1,0,1,1)$. In terms of the operators $B_{1,i}=\Lambda_{1,i}+U_i$ and $M=t_{1,1}B_{1,1}+t_{1,2}B_{1,2}$, the system (4.1) is expressed as

$$[\partial_{1,1}-B_{1,1},\partial_{1,2}-B_{1,2}]=0,\quad [d_s-M,\partial_{1,i}-B_{1,i}]=0\quad (i=1,2),$$

with the equations for normalization (2.6). In this section, we show that the sixth Painlevé equation is derived from them.

The operator M is expressed in the form

$$M=\sum_{i=1,2}t_{1,i}\Lambda_{1,i}+\sum_{j=0,1,3,4}\kappa_j\alpha_j^\vee+\eta\alpha_2^\vee+\varphi e_2+\psi f_2,$$

so that

$$\begin{aligned}&\kappa_j=t_{1,1}u_{j,1}+t_{1,2}u_{j,2}\quad (j=0,1,3,4),\quad \eta=t_{1,1}u_{2,1}+t_{1,2}u_{2,2},\\&\varphi=t_{1,1}x_1+t_{1,2}x_2,\quad \psi=t_{1,1}y_1+t_{1,2}y_2.\end{aligned}\tag{4.2}$$

The system (3.1) implies that the variables κ_j $(j=0,1,3,4)$ are independent of $t_{1,i}$ $(i=1,2)$. Then the following lemma is obtained from (3.3), (3.4) and (4.2).

Lemma 4.1. *The variables $u_{j,i}$, x_i, y_i $(i=1,2;j=0,\dots,4)$ are determined uniquely as polynomials in η, φ and ψ with coefficients in $\mathbb{C}(t_{1,i})[\kappa_j]$. Furthermore, the following relation is satisfied*:

$$\eta^2-(\kappa_0+\kappa_1+\kappa_3+\kappa_4)(\eta+1)+\kappa_0^2+\kappa_1^2+\kappa_3^2+\kappa_4^2+\varphi\psi=0.$$

Thanks to this lemma, the system (4.1) can be rewritten into a system of first order differential equations for η and φ; we do not give the explicit formulas here.

We denote by $\mathfrak{n}_+$ the subalgebra of $\mathfrak{g}$ generated by e_j $(j=0,\dots,4)$, and by $\mathfrak{b}_+$ the borel subalgebra of $\mathfrak{g}$ defined by $\mathfrak{b}_+=\mathfrak{h}\oplus\mathfrak{n}_+$. We look for a dependent variable λ such that

$$\begin{aligned}&\widetilde{M}=\exp(-\lambda f_2)M\exp(\lambda f_2)-\exp(-\lambda f_2)d_s(\exp(\lambda f_2))\in\mathfrak{b}_+,\\&\widetilde{B}_{1,i}=\exp(-\lambda f_2)B_{1,i}\exp(\lambda f_2)-\exp(-\lambda f_2)\partial_{1,i}(\exp(\lambda f_2))\in\mathfrak{b}_+\quad (i=1,2),\end{aligned}$$

namely

$$\begin{aligned}&\varphi\lambda^2+(2\eta-\kappa_0-\kappa_1-\kappa_3-\kappa_4)\lambda-\psi=0,\\&\partial_{1,i}(\lambda)+x_i\lambda^2-(u_{0,i}+u_{1,i}-2u_{2,i}+u_{3,i}+u_{4,i})\lambda-y_i=0\quad(i=1,2).\end{aligned}\tag{4.3}$$

Note that the definition of $\widetilde{M}$ and $\widetilde{B}_{1,i}$ arises from the gauge transformation $\Psi\to\Phi$ defined by $\Phi=\exp(-\lambda f_2)\Psi$ on the Lax form (3.5). By Lemma 4.1 together with the system (4.1), we can show that

$$\lambda=-\frac{1}{8\varphi}(8\eta-\alpha_0^2-\alpha_1^2-\alpha_3^2-\alpha_4^2+4),$$

satisfies the equation (4.3), where α_j $(j=0,1,3,4)$ are constants defined by

$$\kappa_j=-\frac{1}{16}(8\alpha_j-\alpha_0^2-\alpha_1^2-\alpha_3^2-\alpha_4^2-4).$$

We also let μ by a dependent variable defined by $\mu=\varphi$ so that

$$\eta=-\lambda\mu+\frac{1}{8}(\alpha_0^2+\alpha_1^2+\alpha_3^2+\alpha_4^2-4),\quad \varphi=\mu.$$

Then the system (4.1) can be regarded as a system of differential equations for variables λ and μ with parameters α_j $(j=0,1,3,4)$.

We now regard the system (4.1) as a system of ordinary differential equations with respect to the independent variable $t=t_{1,1}$ by setting $t_{1,2}=1$. Then the operator $\widetilde{M}$ is written in the form

$$\begin{aligned}\widetilde{M}=&\frac{1}{16}(\alpha_0^2+\alpha_1^2+\alpha_3^2+\alpha_4^2-4)K-\sum_{j=0,1,3,4}\frac{1}{2}(\alpha_j-1)\alpha_j^\vee\\&+F_2e_2-F_0e_0+(t-1)F_1e_1-(t+1)F_3e_3-tF_4e_4\\&+e_{20}-(t-1)e_{21}+(t+1)e_{23}+te_{24},\end{aligned}$$

where

$$F_0=\lambda+t,\quad F_1=\lambda+\frac{t+1}{t-1},\quad F_2=\mu,\quad F_3=\lambda-\frac{t-1}{t+1},\quad F_4=\lambda-\frac{1}{t}.$$

The operator $\widetilde{B}=\widetilde{B}_{1,1}$ is written in the form

$$\begin{aligned}\widetilde{B}=&\widetilde{u}_2K+\sum_{j=0,1,3,4}\widetilde{u}_j\alpha_j^\vee+\widetilde{x}e_2\\&-e_0+(\lambda+1)e_1-(\lambda-1)e_3-\lambda e_4-e_{21}+e_{23}+e_{24},\end{aligned}$$

where $\widetilde{u}_2$ is a polynomial in λ, μ and the other coefficients are given by

$$\begin{aligned}
\Theta_0\widetilde{u}_j &= F_0F_1F_2F_3F_4F_j^{-1} - \sum_{i=0,1,3,4;i\neq j}\frac{1}{2}(\alpha_i+\alpha_j-2)F_0F_1F_3F_4F_i^{-1}F_j^{-1} \\
&\quad -\frac{1}{2}(\alpha_j-1)F_0(F_0-F_1-F_3-F_4) \quad (j=0,1,3,4), \\
\Theta_0\widetilde{x} &= F_0F_2(F_0-F_1-F_3-F_4)+(\alpha_0+\alpha_2-1)F_0 \\
&\quad +(\alpha_1+\alpha_2-1)F_1+(\alpha_3+\alpha_2-1)F_3+(\alpha_4+\alpha_2-1)F_4,
\end{aligned}$$

with

$$\Theta_0=(F_0-F_1)(F_0-F_3)(F_0-F_4), \quad \alpha_2=-\frac{1}{2}(\alpha_0+\alpha_1+\alpha_3+\alpha_4-1).$$

Since $\widetilde{M}$ and $\widetilde{B}$ is obtained from M and $B_{1,1}$ by the gauge transformation, they satisfy

$$\left[d_s-\widetilde{M},\frac{d}{dt}-\widetilde{B}\right]=0.$$

By rewriting this compatibility condition into differential equations for F_j ($j=0,\dots,4$), we obtain the same system as (0.2), (0.3).

Theorem 4.2. *Under the specialization $t_{1,1}=t$ and $t_{1,2}=1$, the system* (4.1) *is equivalent to the sixth Painlevé equation* (0.2), (0.3).

Remark 4.3. *The system* (0.2), (0.3) *can be regarded as the compatibility condition of the Lax pair*

$$d_s(\Phi)=\widetilde{M}\Phi, \quad \frac{d\Phi}{dt}=\widetilde{B}\Phi, \tag{4.4}$$

where $\Phi=\exp(-\lambda f_2)W\exp(\xi)$. Let

$$\Omega=\exp(\omega_1\alpha_1^\vee+\omega_2\alpha_2^\vee+\omega_3\alpha_3^\vee+\omega_4\alpha_4^\vee)\exp(F_0^{-1}e_2)\Phi,$$

where

$$\begin{aligned}
\omega_1&=\frac{1}{2}\log(t^2+2t-1)(t^2+1), \quad \omega_2=\log F_0, \\
\omega_3&=\frac{1}{2}\log(1+2t-t^2)(t^2+1), \quad \omega_4=\frac{1}{2}\log(1+2t-t^2)(t^2+2t-1).
\end{aligned}$$

Then the system (4.4) *is transformed into the Lax pair of the type of* [NY3] *by the gauge transformation $\Phi\to\Omega$.*

Finally, we define the group of symmetries for P_{VI} following [NY2]. Consider the transformations

$$r_i(X) = X\exp(-e_i)\exp(f_i)\exp(-e_i) \quad (i=0,\ldots,4),$$

where

$$X = \exp(\xi)X(0) = W^{-1}Z, \quad \xi = \sum_{i=1,2}\sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$

Under the similarity condition $\mathcal{M}\in\mathfrak{g}_{\geq 0}$, their action on W is given by

$$\begin{aligned} r_i(W) &= \exp(\lambda f_2)\exp\left(\frac{(\alpha_i^\vee|d_s-\widetilde{M})}{(f_i|d_s-\widetilde{M})}f_i\right)\exp(-\lambda f_2)W \quad (i=0,1,3,4),\\ r_2(W) &= W. \end{aligned}$$

We also define

$$r_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \quad (i,j=0,\ldots,4).$$

Then the action of them on the variables λ, μ is described as

$$r_i(F_j) = F_j - \frac{\alpha_i}{F_i}u_{ij} \quad (i,j=0,\ldots,4),$$

where $U=(u_{ij})_{i,j=0}^4$ is the orientation matrix of the Dynkin diagram defined by

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ -1 & -1 & 0 & -1 & -1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that the transformations r_i $(i=0,\ldots,4)$ satisfy the fundamental relations for the generators of the affine Weyl group $W(D_4^{(1)})$.

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